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FAST TRACK COMMUNICATION

Fock space methods and large N

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Abstract
Ideas and techniques (asymptotic decoupling of single-trace subspace, asymptotic operator algebras, duality and role of supersymmetry) relevant in current Fock space investigations of quantum field theories have very simple roles in a class of toy models.

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(Some figures in this article are in colour only in the electronic version)

Hamiltonian methods have a long history in the attempts to understand the bound states spectrum of a strongly interacting relativistic quantum field theory. This is perhaps the hardest and most important problem in a strongly interacting quantum field theory and analytic and numerical efforts were devoted to inventing reliable methods. For several years light-front quantization [1] was a promising approach because of the very different nature of the ground state and some important simplifications of the operators occurring in the Hamiltonian of non-Abelian models.

The analysis of the large-N limit, at t'Hooft coupling fixed, of the models with $SU(N)$ or $U(N)$ invariance provided additional insights, indicating features of string theory in non-Abelian gauge models and the existence of symmetries, conserved quantum numbers and operator algebras occurring only in the asymptotic theory, at $N = \infty$, [2, 3, 6, 12]. Much work was devoted to the evaluation of the spectrum of the Hamiltonian for states in the lowest representation of the group: the singlet sector and the adjoint sector.

A colour-singlet state of n free bosons, with total momentum $\vec{P}$ is represented in the Fock space by a linear superposition of states of the form

$$\mathrm{tr}\left(a^\dagger(\boldsymbol{k}_1)\cdots a^\dagger\left(\boldsymbol{k}_{n_1}\right)\right)\mathrm{tr}\left(a^\dagger(\boldsymbol{p}_1)\cdots a^\dagger\left(\boldsymbol{p}_{n_2}\right)\right)\cdots\mathrm{tr}\left(a^\dagger(\boldsymbol{q}_1)\cdots a^\dagger\left(\boldsymbol{q}_{n_s}\right)\right)|0\rangle \qquad (1)$$

where $\mathrm{tr}()$ denotes the trace on $U(N)$ colour indices, $n = n_1 + n_2 + \cdots + n_s$, $\boldsymbol{P} = \sum \boldsymbol{k}_j + \sum \boldsymbol{p}_j + \cdots + \sum \boldsymbol{q}_j$ and $a^\dagger(\boldsymbol{k})$ are creation operators. These states are called multi-trace states. Matrix-valued operators may be written in terms of the group generators $a(\boldsymbol{k}) = \sum_a \lambda_a a_a(\boldsymbol{k})$, $a^\dagger(\boldsymbol{k}) = \sum_a \lambda_a a_a^\dagger(\boldsymbol{k})$, and all matrix-related coefficients are efficiently evaluated by graphic methods [5, 9]. Several remarkable properties were found in the large-N limit. Operators normally-ordered inside a single trace, that is of the form $\gamma = N^{-(r+s-2)/2}\,\mathrm{tr}(a^\dagger(\boldsymbol{k}_1)\cdots a^\dagger(\boldsymbol{k}_r)a(\boldsymbol{p}_1)\cdots a(\boldsymbol{p}_s))$, acting on single-trace states generate single-trace states [6], provided $r \geqslant 1$ and $s \geqslant 1$. Then if the Hamiltonian is a linear combination of such operators, the subspace of the Fock space spanned by single-trace states is invariant under the action of the Hamiltonian. These operators and analogous ones involving fermion operators act on single-trace states in a way reminiscent of the coupling of strings. Single-trace states like $|n\rangle = N^{-n/2}n^{-1/2}\,\mathrm{tr}(a^\dagger(\boldsymbol{k}_1)\cdots a^\dagger(\boldsymbol{k}_n))|0\rangle$ form an orthonormal basis in the colour-singlet and single-trace Fock space [4]. Multi-trace states like in equation (1) provide an orthonormal basis in the singlet-singlet Fock space at $N = \infty$ [4].

Recently G Veneziano and J Wosiek [7] have suggested a supersymmetric model in $D = 1$ spacetime dimension, that is a matrix quantum-mechanical model. It is not surprising that the model is analytically solvable in the large-N limit in several sectors of the Fock space and reliable numerical evaluations may be performed in other sectors. Several features of the bound states are very interesting.

The goal of this paper is to use results derived in a recent analysis of the bosonic sector of the model [8] and in a simple generalization presented here, to comment on the properties of V–W Hamiltonian with a view to the role they may have in models in more realistic spacetime dimension.

In the bosonic sector the Hamiltonian of the V–W model is

$$H = \mathrm{tr}(a^\dagger a + g(a^{\dagger 2}a + a^\dagger a^2) + g^2 a^{\dagger 2}a^2), \qquad \lambda = g^2 N. \tag{2}$$

In the single trace sector of the singlet states, for large N, the Hamiltonian is a tridiagonal real symmetric infinite matrix. The bound states spectrum was analytically and numerically solved in the large-N limit for every $\lambda \geqslant 0$ and it presents remarkable features:

- the infinitely many eigenvalues of the discrete spectrum of the model, for $0 \leqslant \lambda < 1$ decrease in a monotonous way as λ increases and all vanish at $\lambda = 1$, where a phase transition occurs. For $\lambda > 1$ one eigenvalue remains at zero energy, it is a new ground state, and the infinitely many eigenvalues increase in a monotonous way as λ increases.
- The non-zero eigenvalues at the pairs of values λ and $1/\lambda$ are related by a duality property

$$\frac{1}{\sqrt{\lambda}}(E_n(\lambda) - \lambda)) = \sqrt{\lambda}\left(E_n\left(\frac{1}{\sqrt{\lambda}}\right) - \frac{1}{\lambda}\right). \tag{3}$$

Let us consider the trivial generalization of equation (2) by allowing two coupling constants[3]

$$H = \mathrm{tr}\left(a^\dagger a + g_3(a^{\dagger 2}a + a^\dagger a^2) + g_4^2 a^{\dagger 2}a^2\right), \qquad \sqrt{\lambda_3} = g_3\sqrt{N}, \qquad \lambda_4 = g_4^2 N. \tag{4}$$

Here too the single-trace sector in the Fock space decouples in the large-N limit and the Hamiltonian is represented by the tridiagonal real symmetric infinite matrix $H(\lambda_3, \lambda_4)$

$$\begin{aligned} H_{j,j+1} &= H_{j+1,j} = \sqrt{\lambda_3}\sqrt{j(j+1)}, \\ H_{j,j} &= (1 + \lambda_4(1 - \delta_{1j}))j, \qquad j = 1, 2, \ldots. \end{aligned} \tag{5}$$

The eigenvalue equation $H\boldsymbol{x} = E\boldsymbol{x}$ is a system of recurrent relations which translates into an easy differential equation for the generating function $G(z) = \sum_1^\infty y_j z^j$ where $x_j = y_j\sqrt{j}$.

$$\sqrt{\lambda_3}\,\omega(z)G'(z) - EG(z) - \left(z\lambda_4 + \sqrt{\lambda_3}\right)G'(0) = 0,$$

where

$$\omega(z) = z^2 + 1 + z(1 + \lambda_4)/\sqrt{\lambda_3}.$$

[3] Since the cubic term in the Hamiltonian (4) is odd under the unitary transformation $a \to -a$ and $a^\dagger \to -a^\dagger$, the spectrum of H is independent of the sign of g_3; hence we assume $g_3 \geqslant 0$ with no loss of generality.

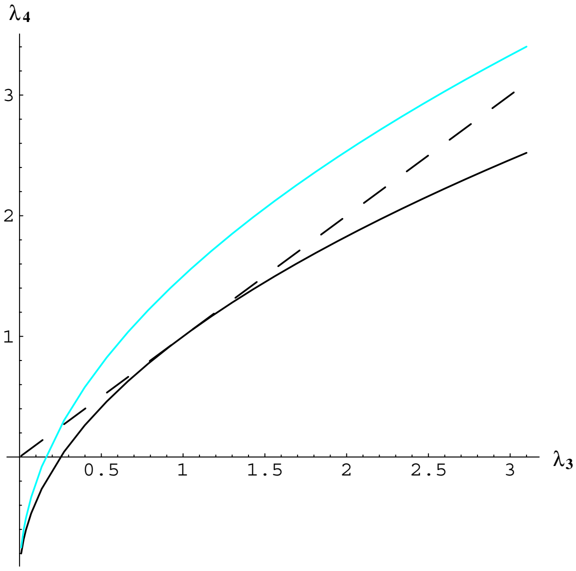

Figure 1. The line $\lambda_3 = \lambda_4$ corresponds to the V–W model.

If $(1+\lambda_4)^2 - 4\lambda_3 > 0, \omega(z)$ has 2 distinct real roots. The closest to the origin $z = 0$ translates into the asymptotic behaviour of the coefficients x_j. The integration constant of the differential equation may be chosen to kill the closest singularity then obtaining an exponentially decreasing sequence x_j, hence normalizable bound states.

The region of the quadrant $\lambda_3 \geqslant 0$ and $\lambda_4 \geqslant -1$ where the spectrum is discrete lies above the parabola of equation $4\lambda_3 = (1+\lambda_4)^2$, shown by the black line in figure 1.

The solution has a compact form in terms of the variables σ and η

$$\sigma = \frac{1+\lambda_4 - \sqrt{(1+\lambda_4)^2 - 4\lambda_3}}{1+\lambda_4 + \sqrt{(1+\lambda_4)^2 - 4\lambda_3}}, \qquad \eta = \frac{\lambda_4(1+\lambda_4) + \lambda_4\sqrt{(1+\lambda_4)^2 - 4\lambda_3} - 2\lambda_3}{2\lambda_4\sqrt{(1+\lambda_4)^2 - 4\lambda_3}}.$$

The eigenvalues E_n of normalizable states are the infinitely many solutions of the equation

$$F(\alpha, 1; 1+\alpha; \sigma) = \eta, \qquad 0 < \sigma < 1,$$

where

$$F(\alpha, 1; 1+\alpha; \sigma) = 1 + \alpha \sum_{k=1}^{\infty} \frac{\sigma^k}{\alpha + k}, \qquad \alpha = -\frac{E}{\sqrt{(1+\lambda_4)^2 - 4\lambda_3}}.$$

It is easy to check that the hypergeometric function $F(\alpha, 1; 1+\alpha; \sigma)$ satisfies the translation identity

$$F(\alpha, 1; 1+\alpha; \sigma) = 1 + \frac{\alpha\sigma}{1+\alpha} F(\alpha+1, 1; 2+\alpha; \sigma). \tag{6}$$

The plot in figure 2 shows the Hypergeometric function $F(\alpha, 1; 1+\alpha; \sigma)$ versus α for fixed σ and $-6 < \alpha < 2$.

In the space of parameters corresponding to bound states, let us consider the parabolas of equation $\lambda_3 = \bar{\sigma}\left(\frac{1+\lambda_4}{1+\bar{\sigma}}\right)^2$, see figure 1. At each point of the space of coupling constants corresponding to the discrete spectrum there is a unique $\bar{\sigma}$. For a given parabola, it is easy to describe the spectrum as we move along it from $\lambda_4 = 0$ to $\lambda_4 = \infty$. Indeed along this path, the value of η increases in a monotonous way from $\eta = -\infty$ to $\eta = 0$ at the point where it

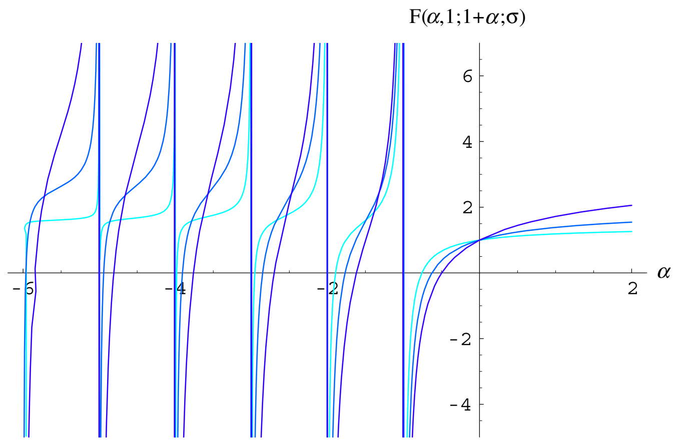

Figure 2. The graph shows $F(\alpha, 1; 1+\alpha; \sigma)$ for 3 values $\sigma = 0.3, 0.5, 0.7$ depicted with increasingly dark colour.

first crosses the line $\lambda_4 = \lambda_3$, next it reaches $\eta = 1$ at the second crossing with the same line, and continues increasing up to its asymptotic limit $\eta = 1/(1-\bar{\sigma}^2)$.

At $\eta = -\infty$ the eigenvalue equation has infinitely many solutions of the form $\alpha_n = -n+\epsilon_n$ where $n = 1, 2, \ldots$ and ϵ_n are small positive numbers. That is $E_n = n(1-\bar{\sigma})/(1+\bar{\sigma}) - \epsilon_n$. As η increases all roots α_n move in a monotonous way to the right, that is each E_n decreases. At $\eta = 1$, the value of $\alpha_1 = 0$ and $E_1 = 0$. This bound state becomes degenerate with the vacuum state $|0\rangle$ of the Fock space, which has $E_0 = 0$. As η increases beyond $\eta = 1$, α_1 increases to positive values and E_1 has *increasingly negative* values[4].

At the first crossing $\lambda_3 = \lambda_4 = \lambda < 1, \eta = 0, \bar{\sigma} = \lambda$ and we may consider the roots $\alpha_n + 1$ of the eigenvalue equation $F(\alpha_n + 1, 1; \alpha_n + 2; \lambda) = 0$. Because of the translation identity equation (6) they are simply related to the roots α_n of the eigenvalue equation at the second crossing $\lambda_3 = \lambda_4 = \lambda > 1, \eta = 1, \bar{\sigma} = 1/\lambda$, which is $F(\alpha_n, 1; \alpha_n + 1; 1/\lambda) = 1$. This is the V–W duality of equation (3).

This picture looks very different from the V–W spectrum but, of course, it is fully compatible: in the V–W model the unique coupling moves along the line $\lambda_4 = \lambda_3$ and it touches at $\lambda = 1$ the boundary of normalizable eigenstates.

We now briefly describe the solution of the 2-couplings model by use of a non-compact Lie algebra which arises in the large-N limit.

Let us consider the Hamiltonian $\tilde{H}(\alpha, \beta)$

$$\tilde{H}(\alpha, \beta) = \alpha D + \tfrac{1}{2}\beta(J_+ + J_-),$$
$$D_{ij} = j\delta_{ij}, \qquad (J_+)_{ij} = \sqrt{j(j+1)}\delta_{i,j+1}, \qquad J_- = (J_+)^\dagger$$
$$J_\pm = J_x \pm iJ_y, \qquad \alpha = 1 + \lambda_4, \qquad \beta = 2\sqrt{\lambda_3}.$$

The Hamiltonian $\tilde{H}(\alpha, \beta)$ differs from the asymptotic Hamiltonian $H(\lambda_3, \lambda_4)$ given in equation (5) only for one matrix element $\tilde{H}_{11} = H_{11} + \lambda_4$. One easily computes the commutators $[D, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = -2D$, showing that $\{D, J_+, J_-\}$ form a basis for the Lie algebra $SO(2,1)$ in the degenerate Bargmann's discrete series representation $\mathcal{D}^n_+$, $(n = 0)$,characterized by a vanishing Casimir operator. Hence the generator J_x has a continuous spectrum filling the whole real axis [11].

4 Some readers may appreciate understanding the model without need of extensive numerical work.

Previous results about the spectrum can be re-derived as follows: if $\alpha > \beta$, one obtains

$$\frac{1}{\sqrt{\alpha^2-\beta^2}}\tilde{H} = \cosh y D + \sinh y J_x, \qquad \cosh y = \frac{\alpha}{\sqrt{\alpha^2-\beta^2}}$$

and a unitary operator U (a boost) exists such that $U\frac{1}{\sqrt{\alpha^2-\beta^2}}\tilde{H}U^{-1} = D$ and the spectrum of $\tilde{H}$ is simply $E_n = n\sqrt{\alpha^2-\beta^2}$.

If $\alpha < \beta$, upon writing

$$\frac{1}{\sqrt{\beta^2-\alpha^2}}\tilde{H} = \sinh y D + \cosh y J_x, \qquad \cosh y = \frac{\beta}{\sqrt{\beta^2-\alpha^2}}$$

a unitary operator boosting $\tilde{H}$ to J_x exists, hence $\tilde{H}$ has a continuous spectrum. On the border $\alpha = \beta$, $\tilde{H}$ coincides with a light-cone generator which has a continuous (positive) spectrum [11].

Finally the spectrum of the asymptotic Hamiltonian $H(\lambda_3, \lambda_4)$ may be computed from the spectrum of $\tilde{H}$ by the method outlined in the appendix of [8], based on an exact perturbation formula (rank-1 perturbation).

Notice that the operators closing the $SO(2, 1)$ algebra represent the restriction to the single-trace states of more general operators acting on general singlet states. For instance $J_+ \to \mathrm{tr}(a^{\dagger 2}a)/\sqrt{N}$. The operators (H, J_+, J_-) close a Lie algebra up to terms of order $1/N$, hence the spectra can be discussed as arising from a dynamical symmetry breaking.

Let us summarize our conclusions:

- Hamiltonian models where each operator has the form $\mathrm{Tr}[(a^\dagger)^n a^m]$ with $n \geqslant 1$ and $m \geqslant 1$ leave each sector of k-trace states invariant in the large-N limit. By representing the Hamiltonian in the basis of single-trace states, one obtains a real symmetric fixed-width band matrix. The eigenvalue equation is a system of recurrent equations which usually allows analytic solution. Still multi-trace singlet sectors are not irrelevant because the mass of bound states in these sectors is similar to that of the single-trace states. As indicated in [8] the V–W bosonic Hamiltonian may be evaluated in each sector obtaining the same eigenvalues, therefore changing (in infinite way) the degeneracy of eigenvalues found in the single-trace analysis.
- In the simple models in $D = 1$ supersymmetry is not necessary for the asymptotic decoupling of the single-trace sector nor for the exact asymptotic solution of the model. However the duality property of the spectrum has a striking simple form only on the susy line $\lambda_3 = \lambda_4$.
- Light-front quantization seems relevant for realistic models, that is in space-time dimension $1 < D \leqslant 4$, because it makes possible to represent a local Hamiltonian in a partial normally ordered form [12] such that the single-trace sector of Fock space may be asymptotically invariant. This is a practical necessity for the Tamm-Dancoff approach. Exact or approximate algebras of the operators which occur in the asymptotic Hamiltonian provide a precious tool for the understanding of the spectrum.
- When considering a local quantum mechanical Hamiltonian $H = \mathrm{Tr}[p^2 + x^2 + V(x)]$ in the large-N limit, the Fock space methods seem inappropriate: one cannot avoid operator terms that couple the single-trace states to multiple-trace states in leading order. This makes any evaluation restricted to the single-trace Fock states totally unreliable [8, 10].
- At large N a new dynamical symmetry shows up, simplifying the calculation of the spectrum. The interplay of this symmetry with supersymmetry as in V–W model deserves further study.

We add a last comment to indicate that the bosonic two-couplings model discussed in this communication is the asymptotic generic form of infinitely many Hamiltonians.

Let us consider the operators $A_j, A_j^\dagger, D_j, j = 1, 2, \ldots$

$$A_j^\dagger = \frac{1}{N^{j-1/2}} \mathrm{tr}(a^\dagger (a^\dagger a)^j), \qquad A_j = \frac{1}{N^{j-1/2}} \mathrm{tr}((a^\dagger a)^j a),$$
$$D_j = \frac{1}{N^{j-1}} \mathrm{tr}((a^\dagger a)^j).$$

It is easy to verify that asymptotically they leave the sector of the single-trace states invariant

$$A_j|n\rangle \sim \sqrt{n(n-1)}|n-1\rangle + O(1/N),$$
$$A_j^\dagger|n\rangle \sim \sqrt{n(n+1)}|n+1\rangle + O(1/N),$$
$$D_j|n\rangle \sim n|n\rangle + O(1/N).$$

Then all the Hamiltonians

$$H = \mathrm{tr}(a^\dagger a) + \sum_{j\geqslant 1} g_{j,3}\, \mathrm{tr}\big(A_j^\dagger + A_j\big) + \sum_{j\geqslant 2} g_{j,4}\, \mathrm{tr}(D_j)$$

asymptotically leave the sector of the single-trace states invariant and in this sector are all represented by the tridiagonal real symmetric infinite matrix $H(\lambda_3, \lambda_4)$ in equation (5) with $\sqrt{\lambda_3} = \sqrt{N} \sum_j g_{j,3}$ and $\lambda_4 = N \sum_j g_{j,4}$. The sums may be finite or infinite.

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